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FEEDBACK CONTROL OF LOCALLY LUMPED STABILIZERS FOR DAMPING MEMBRANE OSCILLATIONS WITH OPTIMIZATION OF POINTS OF STABILIZERS PLACEMENT AND POINTS OF MEASURING

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Abstract. An approach to the synthesis of controlling of lumped sources for distributed systems with feedback is proposed in the paper. We solve the problem of damping membrane oscillations by using point stabilizers, and optimize the following parameters: 1) the points of placement of stabilizers; 2) the points of measurement membrane state; 3) parameters of linear feedback, which define the relationship between the measurements of membrane state and the modes of function of stabilizers. The formulas of the functional gradient by the optimized parameters are obtained.

Keywords: thin membrane, oscillation, synthesis of control, lumped source, the neighborhood of the point of control, loaded differential equation, gradient projection method.
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1 Introduction

As it is known, the problems of optimal control distributed parameters objects with feedback (Butkovskiy, 1984; Aida-zade & Abdullayev, 2012; Ray, 2002) have been studied less than the problems of control the objects with lumped parameters (Polyak, 2019; Krasovskiy, 1987; Yegorov, 2004). This is due, firstly, to the complexity of the technical implementation of such systems, which require obtaining current information about the state of an object (process) at all its points. Secondly, there are problems associated with both solving problems of structural and parametric identification of mathematical models of controlled objects, and developing effective numerical methods and algorithms for solving corresponding mathematical problems.

In recent years, in connection with the development of information and computer technologies and measuring instruments, interest in creating automatic control and regulation systems for complex objects with distributed parameters described by various types of functional equations with initial-boundary conditions has been grown (Butkovskiy, 1984; Aida-zade & Abdullayev, 2012; Ray, 2002).

This paper is devoted to the presentation of the approach to the synthesis of control actions on the process of stabilization of oscillations of a thin homogeneous membrane. It is assumed that the oscillations arose as a result of lumped influences on the membrane at the initial moment of time. The values of the stabilizing effects on the membrane from the side of the stabilizers that were installed at its various points are assigned depending on the measured states of the membrane at neighborhood of the control points.

For the synthesis of control actions, it is proposed to use linear feedback taking into account the results of current measurements. Optimized parameters for the problem under consideration statement are: 1) linear feedback parameters; 2) the coordinates of the stabilizer placement at the membrane; 3) the coordinates of the points of measurement (control) of the state of the membrane. Formulas for the gradient components of the objective functional of the problem are obtained on the space of parameters under optimization.

At the end of the paper numerical experiments are presented which are carried out the analysis of the effect of errors of measurements at measured points for the process of stabilization of a membrane.

The approach presented here can be used for control systems with linear feedback in other technological processes and objects with the distributed parameters which are described by other forms and types of initial-boundary value problems of partial differential equations.

2 Statement of the Problem

The problem of damping the transverse oscillations of a thin uniform membrane of a given shape fixed along the boundary is considered. It is assumed that the oscillations arise as a result of simultaneous effects of external sources at the initial moment of time at the neighborhood of some membrane points θ^ν, $\nu = 1, \dots, L$. The oscillations are dumped by the stabilizers (dampers) acting at the neighborhood of the membrane points η^i, $i = 1, \dots, N_c$, at the neighborhood of the discrete given time points τ_s, $s = 1, \dots, N_t$. To form stabilizers operating modes, data on the measurement results that are obtained by instruments, as well as measurement results at the neighborhood of the points ξ^j, $j = 1, \dots, N_o$.

This process for $t > 0$ can be described by the following initial-boundary value problem (Tikhonov, 1977):

$$u_{tt}(x,t) = a^2 \mathcal{L} u(x,t) - \lambda u_t(x,t) + \sum_{s=1}^{N_t} \delta(t; \mathcal{O}_{\varepsilon_t}(\tau_s)) \sum_{i=1}^{N_c} \vartheta_s^i \delta(x; \mathcal{O}_{\varepsilon_x}(\eta^i)), \tag{1}$$

$$x = (x_1, x_2) \in \Omega \subset \mathrm{R}^2, \quad t \in (0, T_f],$$

$$u(x,0) = 0, \quad u_t(x,0) = \sum_{\nu=1}^{L} q^\nu \delta(x; \mathcal{O}_{\varepsilon_x}(\theta^\nu)), \quad x \in \Omega, \tag{2}$$

$$u(x,t) = 0, \quad x \in \Gamma, \quad t \in (0, T_f]. \tag{3}$$

Here $u(x,t)$ is a function that determines the amount of membrane displacement at point $x \in \Omega$ at time t during its oscillation; $\mathcal{L} = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$; a^2, $\lambda \geq 0$ are the preset constants defined by the physical properties of the membrane and the environment in which it is located; Γ is almost everywhere the smooth boundary of the domain Ω occupied by the membrane, q^ν is the intensity (power) of an external source concentrated in the neighborhood of the membrane point $\theta^\nu = (\theta_1^\nu, \theta_2^\nu) \in \Omega$, $\nu = 1, \dots, L$, the number of which is L; $\vartheta = (\vartheta_1^1, \dots, \vartheta_1^{N_c}, \dots, \vartheta_{N_t}^1, \dots, \vartheta_{N_t}^{N_c}) \in \mathrm{R}^{N_t N_c}$ is the vector that defines the control actions of stabilizers at the neighborhood of the points $\eta^i = (\eta_1^i, \eta_2^i) \in \Omega$, $i = 1, \dots, N_c$, $\eta = (\eta^1, \dots, \eta^{N_c})$, $\tau = (\tau_1, \dots, \tau_{N_t})$ are given time points, at the neighborhood of which there was an effect of dampers, $\tau_s > \tau_{s-1} > 0$, $s = 1, \dots, N_t$, $\tau_0 = 0$, $\tau_{N_t} = T_f$; N_t , the amount of which is N_t; T_f is the given duration of the process control time.

Continuously differentiable on $x \in \Omega$ function $\delta\left(x; \mathcal{O}_{\varepsilon_x}(\tilde{\eta})\right))$ defines distribution of intensity of the sources at the neighborhood $\mathcal{O}_{\varepsilon_x}(\tilde{\eta}))$ of the location point $\tilde{\eta} \in \Omega$. It has the following properties:

$$\delta\left(x; \mathcal{O}_{\varepsilon_x}(\tilde{\eta})\right) = \begin{cases} \neq 0, & \text{if} \quad x \in \mathcal{O}_{\varepsilon_x}(\tilde{\eta}), \\ = 0, & \text{if} \quad x \notin \mathcal{O}_{\varepsilon_x}(\tilde{\eta}), \end{cases}$$

$$\iint\limits_{\Omega} \delta\left(x; \mathcal{O}_{\varepsilon_x}(\tilde{\eta})\right) dx = \iint\limits_{\mathcal{O}_{\varepsilon_x}(\tilde{\eta})} \delta\left(x; \mathcal{O}_{\varepsilon_x}(\tilde{\eta})\right) dx = 1, \quad \tilde{\eta} \in \Omega_{\varepsilon_x}.$$

Ω_{ε_x} is ε_x interior of Ω, i.e. each point $x \in \Omega_{\varepsilon_x}$ is distant from the boundary of the domain Ω not less than by ε_x. The boundary of the neighborhood $\mathcal{O}_{\varepsilon_x}(\tilde{\eta})$ is denoted by $\Gamma_{\varepsilon_x}(\tilde{\eta})$). Similarly, continuous on $t \in [0, T_f]$ the function $\delta\left(t; \mathcal{O}_{\varepsilon_t}(\tilde{\tau})\right)$ determines distribution of intensity of the sources in the neighborhood $\mathcal{O}_{\varepsilon_x}(\tilde{\eta})$ of the action time moment $\tilde{\tau} \in \mathcal{O}_{\varepsilon_t}(\tilde{\tau}) \subset [\varepsilon_t, T_f - \varepsilon_t]$, more over:

$$\delta\left(t; \mathcal{O}_{\varepsilon_t}(\tilde{\tau})\right) = \begin{cases} \neq 0, & \text{if} \quad t \in \mathcal{O}_{\varepsilon_t}(\tilde{\tau}) \subset [\varepsilon_t, T_f - \varepsilon_t], \\ = 0, & \text{if} \quad t \notin \mathcal{O}_{\varepsilon_t}(\tilde{\tau}) \subset [\varepsilon_t, T_f - \varepsilon_t], \end{cases}$$

$$\int\limits_0^{T_f} \delta\left(t; \mathcal{O}_{\varepsilon_t}(\tilde{\tau})\right) dt = \int\limits_{\mathcal{O}_{\varepsilon_t}(\tilde{\tau})} \delta\left(t; \mathcal{O}_{\varepsilon_t}(\tilde{\tau})\right) dt = 1.$$

From the properties of the functions $\delta\left(x; \mathcal{O}_{\varepsilon_x}(\tilde{\eta})\right)$ and $\delta\left(t; \mathcal{O}_{\varepsilon_t}(\tilde{\tau})\right)$ which are involved in the differential equation (1) and the initial condition (2), it follows that measurements and effect at time moment and at points of the membrane are not carried out instantaneously or point-wise, but they have a temporal and spatial distribution in sufficiently small its neighborhood. First, it is explained by practical considerations, since real measurements can not be instantaneous in time and point-wise in phase space. Secondly, when ε_x and ε_t approaches to zero, the functions $\delta\left(x; \mathcal{O}_{\varepsilon_x}(\tilde{\eta})\right)$ and $\delta\left(t; \mathcal{O}_{\varepsilon_t}(\tilde{\tau})\right)$ i.e. the corresponding Dirac functions (Butkovskiy, 1984; Lions, 1971), would be significantly complicate for mathematical evaluation which is presented in below and lead to the necessity of using functional spaces and notions for the solution of the initial-boundary value problem (1)−(3) in a generalized meaning. Thirdly, for the conducted numerical experiments on the test problem, the results of which are given in the article, by using of the functions $\delta\left(x; \mathcal{O}_{\varepsilon_x}(\tilde{\eta})\right)$ and $\delta\left(t; \mathcal{O}_{\varepsilon_t}(\tilde{\tau})\right)$, are more natural for the numerical approximation of the problem.

Here, the considered initial-boundary problem for the hyperbolic type equation is understood in the classical sense.

Let the values of powers of the sources of oscillations q^ν and the places of their locations θ^ν, $\nu = 1, \ldots, L$, are not known exactly. The sets of available values of q^ν are given:

$$Q^\nu = \{\ q \in \mathrm{R}: \ \underline{q^\nu} \leq q \leq \overline{q^\nu}\ \}, \quad \nu = 1, \ldots, L, \quad Q = Q^1 \times \cdots \times Q^L, \tag{4}$$

and functions of distribution density of their values $\rho_{Q^\nu}(q) \geq 0$, $\underline{q^\nu} \leq q \leq \overline{q^\nu}$ are such that

$$\int\limits_{Q^\nu} \rho_{Q^\nu}(q) dq = 1, \quad \nu = 1, \ldots, L.$$

The points θ^ν of the possible locations of sources of external actions are determined by the sets

$$\Theta^\nu \subset \Omega, \quad \nu = 1, \ldots, L, \quad \Theta = \Theta^1 \times \cdots \times \Theta^L, \tag{5}$$

with given distribution density functions $\rho_{\Theta^\nu}(\theta) \geq 0$ such that

$$\iint\limits_{\Theta^\nu} \rho_{\Theta^\nu}(\theta) d\theta = 1, \quad \nu = 1, \ldots, L.$$

The values ϑ_s^i which determine the control powers of the actions and the places of their location η^i are optimize parameters of the considered process of controlling oscillation damping. They satisfy the restrictions:

$$\underline{\vartheta^i} \leq \vartheta_s^i \leq \overline{\vartheta^i}, \quad i = 1, \ldots, N_c, \quad s = 1 \ldots, N_t, \tag{6}$$

$$\eta^i \in \mathcal{O}_{\varepsilon_x}\left(\eta^i\right) \subset \Omega_c^i \subset \Omega, \quad i = 1, \ldots, N_c. \tag{7}$$

These restrictions are based on technical and technological considerations. In (7) Ω_c^i are given closed sub-domains in which stabilizers can be installed; $\underline{\vartheta}^i$, $\overline{\vartheta^i}$ are given $i = 1, \ldots, N_c$.

It is known that the there is classical solution $u(x,t)$ as a meaning of Tikhonov (1977) for the initial-boundary value problem (1)−(3) with the given external and control impulsive effect respectively q^ν, $\nu = 1, \ldots, L$, and ϑ_s^i, $i = 1, \ldots, N_c$.

The considered task of controlling the process of dumping of membrane oscillation for a given time T_f consists in determination the values of powers of the effects of the dumpers of oscillation ϑ and their locations η that satisfy the above conditions and minimize the following functional:

$$J(\vartheta, \eta) = \int\limits_Q \iint\limits_\Theta I(q, \theta; \vartheta, \eta) \rho_\Theta(\theta) \rho_Q(q) d\theta dq, \tag{8}$$

$$I(\vartheta, \eta; q, \theta) = \int\limits_{T_f}^{T_1} \iint\limits_\Omega \mu(x) \big[u(x, t; \vartheta, \eta, q, \theta)\big]^2 dx dt + \mathcal{R}(\vartheta, \eta, \varepsilon), \tag{9}$$

$$\mathcal{R}(\vartheta, \eta, \varepsilon) = \varepsilon_1 ||\vartheta(t) - \hat{\vartheta}(t)||^2_{L_2^{N_c}[0,T_1]} + \varepsilon_2 ||\eta - \hat{\eta}||^2_{\mathrm{R}^{2N_c}},$$

$$q = \{q^1, \ldots, q^L\}, \quad \rho_Q(q) = \rho_{Q^1}(q^1) \cdots \rho_{Q^L}(q^L), \quad dq = dq^1 \cdots dq^L,$$

$$\theta = \{\theta^1, \ldots, \theta^L\}, \quad \rho_\Theta(\theta) = \rho_{\Theta^1}(\theta^1) \cdots \rho_{\Theta^L}(\theta^L), \quad d\theta = d\theta^1 \cdots d\theta^L.$$

Here, the function $u(x,t) = u(x, t; \vartheta, \eta, q, \theta)$ is the solution of the initial-boundary value problem (1)−(3) of given external effect with the power q^ν at the initial moment of time, $\nu = 1, \ldots, L$, and damping modes ϑ; $\mu(x) \geq 0$ is a weight function that determines the value of damping of oscillation at the point $x \in \Omega$ of the membrane. The second term in (8), (9) respond to regularize of the functional, ε_1, ε_2, $\hat{\vartheta} \in \mathrm{R}^{N_t N_c}$, $\hat{\eta} \in \mathrm{R}^{2Nc}$ − regularization parameters.

The given value of ΔT defines time duration of the interval $[T_f, T_1]$, where $[T_f, T_1]$, $T_1 = T_f + \Delta T$, in which it must be observe steady-state of the membrane on this interval. The functional (8) which defines state of a membrane on a time interval $[T_f, T_1]$ estimates quality of the controlling parameters ϑ and η in the controlling process of damping oscillation over time interval $t \in [0, T_f]$ with a average parameters of external point-wise effect of q and ϑ they satisfy the constraints (4) and (5).

Suppose that at the points of the membrane $\xi^j = (\xi_1^j, \xi_2^j) \in \Omega$, $j = 1, \ldots, N_o$, sensors are installed. These sensors measure the integral values of the membrane displacement at the neighborhood of these points and time points $\tau_s \in (0, T_f]$, $s = 1, \ldots, N_t$:

$$\hat{u}_s^j = \int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\xi^j)} u(\hat{x}, \hat{t}) \delta(\hat{x}; \mathcal{O}_{\varepsilon_x}(\eta^i)) \delta(\hat{t}; \mathcal{O}_{\varepsilon_t}(\tau_s)) d\hat{x} d\hat{t}, \tag{10}$$

$j = 1, \ldots, N_o$, $s = 1, \ldots, N_t$ and there is possibility of operative establishment admissible stabilization modes ϑ_s^i, $i = 1, \ldots, N_c$, $s = 1, \ldots, N_t$, according to the results of these measurements. Due to the properties of the functions $\delta(\hat{x}; \mathcal{O}_{\varepsilon_x}(\eta^i))$, $\delta(\hat{t}; \mathcal{O}_{\varepsilon_t}(\tau_s))$ formula (10) can be written as follows:

$$\hat{u}_s^j = \int\limits_0^{T_f} \iint\limits_\Omega u(\hat{x}, \hat{t}) \delta(\hat{x}; \mathcal{O}_{\varepsilon_x}(\eta^i)) \delta(\hat{t}; \mathcal{O}_{\varepsilon_t}(\tau_s)) d\hat{x} d\hat{t},$$

Here $\delta\left(x; \mathcal{O}_{\varepsilon_x}\left(\eta^i\right)\right)$, $\delta\left(t; \mathcal{O}_{\varepsilon_t}\left(\tau_s\right)\right)$ are used as weight functions that determine the contribution of the value membrane state at the point $x \in O_{\varepsilon_x}(\xi^j)$, $t \in O_{\varepsilon_t}(\tau_s)$ in general by the measured value $\hat{u}_s^j$.

To assign the current values for the modes of the stabilizers we use the following function that determines the feedback of control actions with the state of the membrane at the neighborhood of observation points:

$$\vartheta_s^i = \sum_{j=1}^{N_o} k^{ij}[u_s^j - z^{ij}] = \sum_{j=1}^{N_o} k^{ij} \left[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\xi^j)} u(\hat{x},\hat{t})\delta\left(\hat{x};\mathcal{O}_{\varepsilon_x}\left(\xi^j\right)\right)\delta\left(\hat{t};\mathcal{O}_{\varepsilon_t}\left(\tau_s\right)\right)d\hat{x}d\hat{t} - z^{ij} \right], \tag{11}$$

$$i = 1,\dots,N_c, \quad s = 1,\dots,N_t.$$

Here $k = \left(\left(k^{ij}\right)\right)$ is the amplification factors matrix; $z = \left(\left(z^{ij}\right)\right)$, z^{ij} is the nominal value of the displacement of the membrane at the point ξ^j relative to the stabilizer installed at the point η^i, $i = 1,\dots,N_c$, $j = 1,\dots,N_o$; k and z are optimized feedback parameters.

If we substitute the formula (11) into equation (1), we obtain:

$$u_{tt}(x,t) = a^2\mathcal{L}u(x,t) - \lambda u_t(x,t) + \sum_{s=1}^{N_t} \delta\left(t;\mathcal{O}_{\varepsilon_t}\left(\tau_s\right)\right) \sum_{i=1}^{N_c} \delta\left(x;\mathcal{O}_{\varepsilon_x}\left(\eta^i\right)\right) \cdot \tag{12}$$

$$\cdot \sum_{j=1}^{N_o} k^{ij} \left[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\xi^j)} u(\hat{x},\hat{t})\delta\left(\hat{x};\mathcal{O}_{\varepsilon_x}\left(\xi^j\right)\right)\delta\left(\hat{t};\mathcal{O}_{\varepsilon_t}\left(\tau_s\right)\right)d\hat{x}d\hat{t} - z^{ij} \right], \quad x \in \Omega.$$

Due to involving in the differential equations the integral values of the desired functions at the neighborhood of some points of the time variable or phase space, many authors call such equations loaded ones (see Nakhushev (2012) and the bibliography in it). In equation (12), the loading are at the neighborhood of the measurement points ξ^j, $j = 1,\dots,N_o$. Studies on the existence, uniqueness of solutions of loaded differential equations for both ordinary and partial derivatives, including numerical methods for solving them, have been studied in some works, such as Alikhanov et al. (2014); Abdullaev & Aida-zade (2014); Aida-zade (2018). Therefore, in this article these issues are not considered.

Let devices for measuring the state of the membrane, based on technical and technological considerations, can be installed not at all points of the membrane, but in some of its given sub-domains:

$$\xi^j \in \mathcal{O}_{\varepsilon_x}\left(\xi^j\right) \subset \Omega_o^j \subset \Omega, \quad j = 1,\dots,N_o, \tag{13}$$

and in practice, as a rule, the sub-domains of the points of placement of stabilizers and measurements of states may be not intersected, i.e.

$$\Omega_c^i \cap \Omega_o^j = \emptyset, \quad i = 1,\dots,N_c, \quad j = 1,\dots,N_o.$$

The main purpose of this article is the synthesis of control parameters for stabilizing the process of membrane oscillations. The problem is to determine the optimal values of the feedback parameters $k \in \mathrm{R}^{N_c N_o}$, $z \in \mathrm{R}^{N_c N_o}$, placement of the points of the measuring ξ and stabilization η for which (6), (7), (12), (2), (3) constraints are satisfied . The total dimension of the finite-dimensional vector of the synthesized parameters, which we denote by $y = (k, z, \xi, \eta)$, is $N = 2(N_c N_o + N_c + N_o)$, i.e. $y \in \mathrm{R}^N$. As we can see, the dimension of the optimized vector is determined mainly by the double product of the number of stabilizers and control points. In practical applications, their number is is not too large and rarely exceeds $5 \div 6$ units, and therefore the dimension of the whole problem is about $60 \div 80$. Such a dimension can be considered acceptable, taking into account the use of modern computer technology, numerical methods and the fact that these problems do not require a real-time solution.

We write the criterion of quality of the control parameters defined by the functional (8) as:

$$J(y) = \int\limits_Q \iint\limits_\Theta I(y;q,\theta)\rho_\Theta(\theta)\rho_Q(q)d\theta dq, \tag{14}$$

$$I(y;q,\theta)=\int_{T_f}^{T_1}\iint_{\Omega}\mu(x)\big[u(x,t;y,q,\theta)\big]^2dxdt+\mathcal{R}(y,\varepsilon), \tag{15}$$

where we use following notations:

$$\mathcal{R}(y,\varepsilon)=\varepsilon_1\|k-\hat{k}\|^2_{\mathrm{R}^{N_cN_o}}+\varepsilon_2\|z-\hat{z}\|^2_{\mathrm{R}^{N_cN_o}}+\varepsilon_3\|\xi-\hat{\xi}\|^2_{\mathrm{R}^{2N_o}}+\varepsilon_4\|\eta-\hat{\eta}\|^2_{\mathrm{R}^{2N_c}}.$$

Here $\varepsilon=(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$, $\varepsilon_i\geq 0$, $i=1,\ldots,4$, $\hat{k}\in\mathrm{R}^{N_cN_o}$, $\hat{z}\in\mathrm{R}^{N_cN_o}$, $\hat{\xi}\in\mathrm{R}^{2N_o}$, $\hat{\eta}\in\mathrm{R}^{2N_c}$ are the regularization parameters, the values of which can be assigned using various known algorithms of the regularization method (Vasil'ev, 2002).

The obtained problem of synthesis the control of stabilization of the oscillations (12), (2), (3), (4)$-$(6), (14), (15) belongs to the class of parametric optimal control problems for distributed systems.

It is interesting when the set of possible values of external influences Q^ν and their displacement points Θ^ν are given discretely,

$$Q^\nu=q^{\nu,i}:\quad i=1,\ldots,N_q^\nu,\quad \nu=1,\ldots,L, \tag{16}$$

$$\Theta^\nu=\theta^{\nu,j}:\quad j=1,\ldots,N_\theta^\nu,\quad \nu=1,\ldots,L, \tag{17}$$

and with probability to obtain these discrete values,

$$p^i_{Q^\nu}=P(q=q^{\nu,i}),\quad i=1,\ldots,N_q^\nu,\quad \nu=1,\ldots,L, \tag{18}$$

$$p^j_{\Theta^\nu}=P(\theta=\theta^{\nu,j}),\quad i=1,\ldots,N_\theta^\nu,\quad \nu=1,\ldots,L. \tag{19}$$

In this case the functional (14) will take the form:

$$J(y)=\sum_{i_1=1}^{N_q^1}\cdots\sum_{i_L=1}^{N_q^L}\sum_{j_1=1}^{N_\theta^1}\cdots\sum_{j_L=1}^{N_\theta^L}I\left(y;q^{1,i_1},\ldots,q^{L,i_L},\theta^{1,j_1},\ldots,\theta^{L,j_L}\right)p^{i_1}_{Q^1}\cdots p^{i_L}_{Q^L}\cdot p^{j_1}_{\Theta^1}\cdots p^{j_L}_{\Theta^L}. \tag{20}$$

In practical applications, there is a case when the feedback with the measurement points of the state (displacement) of the membrane and the effect of stabilizer can be considered only at discrete moments τ_n, $n=1,\ldots,N_m$. The proposed approach can be extended to a formulation of the such problem.

We note the following properties of the optimal synthesis control problem for the process of stabilization of the oscillation.

First, the synthesized control is determined by a finite-dimensional vector $y\in\mathrm{R}^N$. Secondly, in problem optimizes both the coordinates of the stabilizers $\eta^i\in\Omega_c^i\subset\Omega$, and the measurement points $\xi^j\in\Omega_o^j\subset\Omega$, which are determined for all possible intensity values of external point effects and points their concentration. Third, in spite of the fact that the given optimal control problem with respect to equation (1) and functional (15) at the points of location of stabilizer is convex, in general case, the obtaining problem may be non-convex and, therefore, multi-extremal due to nonlinear involving optimized feedback parameters in equation (11). The fourth property of the considering problem is that the it is loaded differential equation of hyperbolic type. The fifth property of the problem is related to the constraint (5) which is posed on the controlling stabilizers. By virtue of (11), constraint (13), which is posed on control, passes to a joint constraint on the phase function $u(x,t)$ at the measurement points ξ^j, $j=1,\ldots,N_o$ and the feedback parameters k, z. The sixth property of the considered problem is that the powers and lumped points external effects on the process are not specified precisely, therefore the objective functional evaluates the quality of the feedback parameters on average by all possible values of external effect. Therefore, it can be expected that the obtained optimal control will be robust with respect to small changes in the initial data of the problem. This property of the considered problem will be illustrated further on the test problem at the final section of the paper.

3 Approach and formulas for the solution of the problem of synthesis of control stabilization of process

First of all, we consider the constraints (6), (7) taking into account the formula (11). We assume that the domain Ω has a simple structure (rectangle, circle, ellipse, etc.), and the operator of projecting points of the space R^2 onto this region has a constructive character.

Taking into account (11) and introducing the notation

$$g_i^0(\tau_s;y)=\frac{\overline{\vartheta^i}+\underline{\vartheta}^i}{2}-\sum_{j=1}^{N_o}k^{ij}\left[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)}\iint\limits_{\mathcal{O}_{\varepsilon_x}(\xi^j)}u(\hat{x},\hat{t})\delta(\hat{x};\mathcal{O}_{\varepsilon_x}(\xi^j))\delta(\hat{t};\mathcal{O}_{\varepsilon_t}(\tau_s))d\hat{x}d\hat{t}-z^{ij}\right],$$

$$i=1,\dots,N_c,\quad s=1,\dots,N_t,$$

we write constraints (6) in compact form:

$$g_i(\tau_s;y)=|g_i^0(\tau_s;y)|-\frac{\overline{\vartheta^i}-\underline{\vartheta}^i}{2}\le 0,\quad i=1,\dots,N_c,\quad s=1,\dots,N_t. \tag{21}$$

To take into account the constraints (21) in the problem of optimizing the feedback parameters y, we use the external penalty method (Vasil'ev, 2002). In the integrand of the objective functional (14), we add the penalty term:

$$J_r(y)=\int\limits_Q\iint\limits_\Theta \tilde{I}_r(y;q,\theta)\rho_\Theta(\theta)\rho_Q(q)d\theta dq, \tag{22}$$

$$\tilde{I}_r(y;q,\theta)=I(y;q,\theta)+rG(y). \tag{23}$$

Here we use following notations:

$$G(y)=\sum_{s=1}^{N_t}\sum_{i=1}^{N_c}[g_i^+(\tau_s;y)]^2,$$

$$g_i^+(\tau_s;y)=\begin{cases}0, & g_i(\tau_s;y)\le 0,\quad i=1,\dots,N_c,\quad s=1,\dots,N_t\\ g_i(\tau_s;y), & g_i(\tau_s;y)>0,\quad i=1,\dots,N_c,\quad s=1,\dots,N_t.\end{cases}$$

In (23), the parameter $r>0$ is a penalty coefficient; and in numerical calculations it is necessary that r tends to $+\infty$.

To take into account the constraints (7), (13) we use the operators of projection on Ω_c^i, Ω_o^j, $i=1,\dots,N_c$, $j=1,\dots,N_o$. In general, for the numerical solution of the problem of synthesis of parameters y, we use the method of projection of the gradient of the penalty functional. The iterative procedure for constructing a minimizing sequence has the following form (Vasil'ev, 2002):

$$y^{m+1}=\mathcal{P}_{(7),(13)}[y^m-\alpha_m\mathrm{grad}J_r(y^m)],\quad m=0,1,2,\dots. \tag{24}$$

Here $\mathrm{grad}J_r(y)$ is the gradient of functional (22) calculated at the point $y\in\mathrm{R}^N$ for a given penalty coefficient r; $\mathcal{P}_{(7),(13)}[\cdot]$ is the operator of projecting of the components of the vector y

(more specifically, its components ξ and η) onto the sub-domains Ω_c^i and Ω_o^j, $i = 1,\ldots,N_c$, $j = 1,\ldots,N_o$; α_m is the value of the step in the direction of the anti-gradient of the penalty function that was projected onto the constraints (7), (13) under which the condition $J_r^{m+1}(y) \leq J_r^m$ must be fulfilled (Vasil'ev, 2002).

We formulate a theorem in which the formulas are given for the gradient components of the functional $J_r(y)$ that are necessary for the implementation of procedure (24). In it the characteristic function $\chi_{[T_f,T_1]}(t)$, which is equal to 0 for $t \notin [T_f,T_1]$ and one for $t \in [T_f,T_1]$ is used.

Theorem 1. *For the components of the gradient of the functional* (14) *by the parameters* $y = (k,z,\xi,\eta) \in R^N$ *of linear feedback* (10), (11), *the formulas take place:*

$$\frac{\partial J_r(y)}{\partial k^{ij}} = \int\limits_Q \iint\limits_\Theta \Big\{ -\sum_{s=1}^{N_t} \Big[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\eta^i)} \psi(x,t)\delta(x;\mathcal{O}_{\varepsilon_x}(\eta^i))\delta(t;\mathcal{O}_{\varepsilon_t}(\tau_s))dxdt + \tag{25}$$

$$+2rg_i^+(\tau_s;y)\mathrm{sgn}(g_i^0(\tau_s;y))\Big]\Big[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\xi^j)} u(x,t)\delta(\hat{x};\mathcal{O}_{\varepsilon_x}(\xi^j))\delta(\hat{t};\mathcal{O}_{\varepsilon_t}(\tau_s))d\hat{x}d\hat{t} - z^{ij}\Big] +$$

$$+2\varepsilon_1(k^{ij} - \hat{k}^{ij})\Big\}\rho_Q(q)\rho_\Theta(\theta)d\theta dq,$$

$$\frac{\partial J_r(y)}{\partial z^{ij}} = \int\limits_Q \iint\limits_\Theta \Big\{ \sum_{s=1}^{N_t} \Big[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\eta^i)} \psi(x,t)\delta(x;\mathcal{O}_{\varepsilon_x}(\eta^i))\delta(t;\mathcal{O}_{\varepsilon_t}(\tau_s))dxdt + \tag{26}$$

$$+2rg_i^+(\tau_s;y)\mathrm{sgn}(g_i^0(\tau_s;y))\Big] k^{ij} + 2\varepsilon_2(z^{ij} - \hat{z}^{ij})\Big\}\rho_Q(q)\rho_\Theta(\theta)d\theta dq,$$

$$\frac{\partial J_r(y)}{\partial \xi_\gamma^j} = \int\limits_Q \iint\limits_\Theta \Big\{ -\sum_{s=1}^{N_t}\sum_{i=1}^{N_c} \Big[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\eta^i)} \psi(x,t)\delta(x;\mathcal{O}_{\varepsilon_x}(\eta^i))\delta(t;\mathcal{O}_{\varepsilon_t}(\tau_s))dxdt + \tag{27}$$

$$+2rg_i^+(\tau_s;y)\mathrm{sgn}(g_i^0(\tau_s;y))\Big] k^{ij}\Big[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\xi^j)} \frac{\partial u(\hat{x},\hat{t})}{\partial \hat{x}_\gamma}\delta(\hat{x};\mathcal{O}_{\varepsilon_x}(\xi^j))\delta(\hat{t};\mathcal{O}_{\varepsilon_t}(\tau_s))d\hat{x}d\hat{t}\Big] +$$

$$+2\varepsilon_3(\xi_\gamma^j - \hat{\xi}_\gamma^j)\Big\}\rho_Q(q)\rho_\Theta(\theta)d\theta dq,$$

$$\frac{\partial J_r(y)}{\partial \eta_\gamma^i} = \int\limits_Q \iint\limits_\Theta \Big\{ -\sum_{s=1}^{N_t} \Big[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\eta^i)} \frac{\partial \psi(x,t)}{\partial x_\gamma}\delta(x;\mathcal{O}_{\varepsilon_x}(\eta^i))\delta(t;\mathcal{O}_{\varepsilon_t}(\tau_s))dxdt\Big] \cdot \tag{28}$$

$$\cdot\sum_{j=1}^{N_o} k^{ij}\Big[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\xi^j)} u(x,t)\delta(\hat{x};\mathcal{O}_{\varepsilon_x}(\xi^j))\delta(\hat{t};\mathcal{O}_{\varepsilon_t}(\tau_s))d\hat{x}d\hat{t} - z^{ij}\Big] +$$

$$+2\varepsilon_4(\eta_\gamma^i - \hat{\eta}_\gamma^i)\Big\}\rho_Q(q)\rho_\Theta(\theta)d\theta dq,$$

$i = 1,\ldots,N_c$, $j = 1,\ldots,N_o$, $\gamma = 1,2$. *The function* $\psi(x,t)$ *is a solution of the following conjugate initial boundary value problem:*

$$\psi_{tt}(x,t) = a^2\mathcal{L}\psi(x,t) + \lambda\psi_t(x,t) - 2u(x,t;y,q,\theta)\chi_{[T,T_1]}(t) + \tag{29}$$
$$+\sum_{s=1}^{N_t}\delta(t;\mathcal{O}_{\varepsilon_t}(\tau_s))\sum_{j=1}^{N_o}\delta(x;\mathcal{O}_{\varepsilon_x}(\xi^j))\sum_{i=1}^{N_c}k^{ij}\cdot$$
$$\cdot\Bigg[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)}\iint\limits_{\mathcal{O}_{\varepsilon_x}(\eta^i)}\psi(\hat{x},\hat{t})\delta(\hat{x};\mathcal{O}_{\varepsilon_x}(\eta^i))\delta(\hat{t};\mathcal{O}_{\varepsilon_t}(\tau_s))d\hat{x}d\hat{t} + 2rg_i^+(\tau_s;y)\mathrm{sgn}(g_i^0(\tau_s;y))\Bigg],$$
$$x\in\Omega, \quad t\in[0,T_1],$$

$$\psi(x,T_1) = 0, \quad \psi_t(x,T_1) = 0, \quad x\in\Omega, \tag{30}$$

$$\psi(x,t) = 0, \quad x\in\Gamma, \quad t\in[0,T_1]. \tag{31}$$

Proof. If we consider that for all $\nu = 1,\ldots,L$ the values of perturbations q^ν and focus of points θ^ν for the location of all external sources are not dependent from each other and do not depend on the optimized synthesized parameter y, then the gradient of the functional (14) will take following form

$$\mathrm{grad}_y J_r(y) = \mathrm{grad}_y\int\limits_Q\iint\limits_\Theta \tilde{I}_r(y;q,\theta)\rho_Q(q)\rho_\Theta(\theta)d\theta dq = \tag{32}$$
$$= \int\limits_Q\iint\limits_\Theta \mathrm{grad}_y\tilde{I}_r(y;q,\theta)\rho_Q(q)\rho_\Theta(\theta)d\theta dq.$$

In this aim, taking into account the given values of external perturbation q^ν and their application points θ^ν, $\nu = 1,\ldots,L$, it is enough to obtain components of the gradient as

$$\mathrm{grad}_y\tilde{I}_r(y;q,\theta) = \left(\frac{\partial\tilde{I}_r(y;q,\theta)}{\partial k};\frac{\partial\tilde{I}_r(y;q,\theta)}{\partial z};\frac{\partial\tilde{I}_r(y;q,\theta)}{\partial \xi};\frac{\partial\tilde{I}_r(y;q,\theta)}{\partial \eta}\right),$$

In the differential equation (12), the optimization parameters y are involved only in the third term of the right-hand side. We denote this term by

$$V(x,t;y\tau) = \sum_{s=1}^{N_t}\delta(t;\mathcal{O}_{\varepsilon_t}(\tau_s))\sum_{i=1}^{N_c}\delta(x;\mathcal{O}_{\varepsilon_x}(\eta^i))\cdot \tag{33}$$
$$\cdot\sum_{j=1}^{N_o}k^{ij}\left[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)}\iint\limits_{\mathcal{O}_{\varepsilon_x}(\xi^j)}u(\hat{x},\hat{t})\delta(\hat{x};\mathcal{O}_{\varepsilon_x}(\eta^i))\delta(\hat{t};\mathcal{O}_{\varepsilon_t}(\tau_s))d\hat{x}d\hat{t} - z^{ij}\right],$$

and we will consider this as aggregated control.

Let us reduce equation (12) to the following form:

$$u_{tt}(x,t) = a^2\mathcal{L}u(x,t) - \lambda u_t(x,t) + V(x,t;y,\tau), \quad x\in\Omega, \quad t\in[0,T_1]. \tag{34}$$

To obtain the desired formulas for the components of the gradient of the objective functional, we use increment method for the Lagrange functional of the problem under consideration, which is obtained by giving increment to the independent parameters y. We construct the Lagrange functional. To do this, we multiply both sides of equation (14) by the still arbitrary unknown function $\psi(x,t)$, the conditions on which will be imposed further in the procedure of the proof.

If we integrate (34) over $x \in \Omega$ with the integration boundary $t \in [0, T_1]$, and adding functional (23) the left side of the obtained relation, we obtain:

$$\tilde{I}_r(y; q, \theta) = I(y; q, \theta) + rG(y) + \\ + \int_0^{T_1} \iint_\Omega \psi(x,t) \left[u_{tt}(x,t) - a^2 \mathcal{L} u(x,t) + \lambda u_t(x,t) - V(x,t;y,\tau) \right] dxdt. \tag{35}$$

It is clear that a change in the values of the parameters y will primarily lead to a change in the value of the aggregated control $V(x,t;y,\tau)$.

Let the control $V(x,t;y,\tau)$ takes increment $\Delta V(x,t;y,\tau)$ by changing the vector of synthesized parameters $y = (k, z, \xi, \eta)$ with the increment $\Delta y = (\Delta k, \Delta z, \Delta \xi, \Delta \eta)$. Denote by $\Delta_k V(x,t;y,\tau)$, $\Delta_z V(x,t;y,\tau)$, $\Delta_\xi V(x,t;y,\tau)$, $\Delta_\eta V(x,t;y,\tau)$ increments of the controlling term $V(x,t;y,\tau)$,) obtained by incrementing Δk, Δz, $\Delta \xi$, $\Delta \eta$ of the corresponding components of the vector y, from which independence the increment $\Delta V(x,t;y,\tau)$ is defined as:

$$\Delta V(x,t;y,\tau) = V(x,t;y+\Delta y,\tau) - V(x,t;y,\tau) = \\ = \Delta_k V(x,t;y,\tau) + \Delta_z V(x,t;y,\tau) + \Delta_\xi V(x,t;y,\tau) + \Delta_\eta V(x,t;y,\tau).$$

Similarly, the increment of the functional $\tilde{I}(y; q, \theta)$, corresponding to the increment $\Delta y = (\Delta k, \Delta z, \Delta \xi, \Delta \eta)$, is determined by the formula:

$$\Delta \tilde{I}_r(y; q, \theta) = \tilde{I}_r(y + \Delta y; q, \theta) - \tilde{I}_r(y; q, \theta) = \\ = \Delta_k \tilde{I}_r(y; q, \theta) + \Delta_z \tilde{I}_r(y; q, \theta) + \Delta_\xi \tilde{I}_r(y; q, \theta) + \Delta_\eta \tilde{I}_r(y; q, \theta),$$

where $\Delta_k \tilde{I}_r(y; q, \theta)$, $\Delta_z \tilde{I}_r(y; q, \theta)$, $\Delta_\xi \tilde{I}_r(y; q, \theta)$, $\Delta_\eta \tilde{I}_r(y; q, \theta)$ are increments of the functional, obtained by increments of the corresponding components of the vector y.

Then the increment of the solution of initial-boundary value problem (12), (2), (3)

$$\Delta u(x,t) = \Delta u(x,t;y) = u(x,t;y+\Delta y) - u(x,t;y),$$

be the solution of the following the initial-boundary value problem:

$$\Delta u_{tt}(x,t) = a^2 \mathcal{L} \Delta u(x,t) - \lambda \Delta u_t(x,t) + \Delta V(x,t;y), \quad x \in \Omega, \quad t \in (0, T_1], \tag{36}$$

$$\Delta u(x,0) = 0, \quad \Delta u_t(x,0) = 0, \quad x \in \Omega, \tag{37}$$

$$\Delta u(x,t) = 0, \quad x \in \Gamma, \quad t \in (0, T_1]. \tag{38}$$

We obtain the increment formula $\Delta \tilde{I}(y; q, \theta)$ for the functional (35), corresponding to the increment of the aggregate control $\Delta V(x,t;y,\tau)$ with accuracy small of the second order which is written in the form:

$$\Delta \tilde{I}_r(y; q, \theta) = \int_0^{T_1} \iint_\Omega \left[\frac{\partial \tilde{I}_r(y; q, \theta)}{\partial k} \Delta k + \frac{\partial \tilde{I}_r(y; q, \theta)}{\partial z} \Delta z + \frac{\partial \tilde{I}_r(y; q, \theta)}{\partial \xi} \Delta \xi + \frac{\partial \tilde{I}_r(y; q, \theta)}{\partial \eta} \Delta \eta \right] dxdt + \\ + o\left(\|\Delta_k\|_{R^{N_c N_o}} \right) + o\left(\|\Delta_z\|_{R^{N_c N_o}} \right) + o\left(\|\Delta_\xi\|_{R^{2N_o}} \right) + o\left(\|\Delta_\eta\|_{R^{2N_c}} \right).$$

For this, we use known calculations (integration by parts, grouping), similar to those which is given, for example, in Vasil'ev (2002). Then, we get

$$\Delta\tilde{I}_r(y;q,\theta) = 2\int\limits_{T_f}^{T_1}\iint\limits_{\Omega} \mu(x)u(x,t)\Delta u(x,t)dxdt+ \tag{39}$$

$$+\int\limits_{0}^{T_1}\iint\limits_{\Omega}\left[\psi_{tt}(x,t) - a^2\mathcal{L}\psi(x,t) - \lambda\psi_t(x,t)\right]\Delta u(x,t)dxdt+$$

$$+\iint\limits_{\Omega}\psi(x,T_1)\Delta u_t(x,T_1)dx - \iint\limits_{\Omega}\left[\psi_t(x,T_1) - \lambda\psi(x,T_1)\right]\Delta u(x,T_1)dx-$$

$$-a^2\int\limits_{0}^{T_1}\iint\limits_{\Omega}\psi(x,t)\frac{\partial\Delta u(x,t)}{\partial n}dxdt + a^2\int\limits_{0}^{T_1}\iint\limits_{\Omega}\frac{\partial\psi(x,t)}{\partial n}\Delta u(x,t)dxdt-$$

$$-\iint\limits_{\Omega}\psi(x,0)\Delta u_t(x,0)dx + \iint\limits_{\Omega}\left[\psi_t(x,0) - \lambda\psi(x,0)\right]\Delta u(x,0)dx-$$

$$-\int\limits_{0}^{T_1}\iint\limits_{\Omega}\psi(x,t)\Delta V(x,t;y)dxdt + r\Delta G(y) + 2\varepsilon_1\langle k-\hat{k},\Delta k\rangle + 2\varepsilon_2\langle z-\hat{z},\Delta z\rangle+$$

$$+2\varepsilon_3\langle\xi-\hat{\xi},\Delta\xi\rangle + 2\varepsilon_4\langle\eta-\hat{\eta},\Delta\eta\rangle + o(\|\Delta u(x,t)\|_{L_2(\Omega\times[0,T_1])}) + o(\|\Delta y\|_{R^N}).$$

Since the function $\psi(x,t)$ is arbitrary, we require it to be a solution to the conjugate initial-boundary value problem (29)−(31). Then, by virtue of (36)−(38) from (39), we will have

$$\Delta\tilde{I}_r(y;q,\theta) = -\int\limits_{0}^{T_1}\iint\limits_{\Omega}\psi(x,t)\Delta V(x,t;y)dxdt + r\Delta G(y) + 2\varepsilon_1\langle k-\hat{k},\Delta k\rangle+$$

$$+2\varepsilon_2\langle z-\hat{z},\Delta z\rangle + 2\varepsilon_3\langle\xi-\hat{\xi},\Delta\xi\rangle + 2\varepsilon_4\langle\eta-\hat{\eta},\Delta\eta\rangle+$$

$$+o(\|\Delta u(x,t)\|_{L_2(\Omega\times[0,T_1])}) + o(\|\Delta y\|_{R^N}).$$

It is clear that,

$$\int\limits_{0}^{T_1}\iint\limits_{\Omega}\psi(x,t)\,\Delta V(x,t;y)\,dxdt =$$

$$=\int\limits_{0}^{T_1}\iint\limits_{\Omega}\psi(x,t)\Big(\Delta V_k(x,t;y),\Delta V_z(x,t;y),\Delta V_\xi(x,t;y),\Delta V_\eta(x,t;y)\Big)dxdt.$$

Taking into account the notation (33), the following formulas hold:

$$\int\limits_{0}^{T_1}\iint\limits_{\Omega}\psi(x,t)\,\Delta_k V(x,t;y,\tau)\,dxdt = \sum_{s=1}^{N_t}\sum_{i=1}^{N_c}\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)}\iint\limits_{\mathcal{O}_{\varepsilon_x}(\eta^i)}\psi(x,t)\delta(x;\mathcal{O}_{\varepsilon_x}(\eta^i))\delta(t;\mathcal{O}_{\varepsilon_t}(\tau_s))dxdt\cdot$$

$$\cdot\sum_{j=1}^{N_o}\left[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)}\iint\limits_{\mathcal{O}_{\varepsilon_x}(\xi^j)}\psi(\hat{x},\hat{t})\delta(\hat{x};\mathcal{O}_{\varepsilon_x}(\xi^j))\delta(\hat{t};\mathcal{O}_{\varepsilon_t}(\tau_s))d\hat{x}d\hat{t} - z^{ij}\right]\Delta k^{ij} + o(\|\Delta k\|_{\mathrm{R}^{N_cN_o}}),$$

$$\int_0^{T_1} \iint_\Omega \psi(x,t)\,\Delta_z V(x,t;y,\tau)\,dxdt = \tag{40}$$

$$= -\sum_{s=1}^{N_t}\sum_{i=1}^{N_c} \int_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint_{\mathcal{O}_{\varepsilon_x}(\eta^i)} \psi(x,t)\delta(x;\mathcal{O}_{\varepsilon_x}(\eta^i))\delta(t;\mathcal{O}_{\varepsilon_t}(\tau_s))dxdt \sum_{j=1}^{N_o} k^{ij}\Delta z^{ij} + o(\|\Delta z\|_{\mathrm{R}^{N_c N_o}}),$$

$$\int_0^{T_1} \iint_\Omega \psi(x,t)\,\Delta_\xi V(x,t;y,\tau)\,dxdt = \tag{41}$$

$$= -\sum_{s=1}^{N_t}\sum_{i=1}^{N_c} \int_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint_{\mathcal{O}_{\varepsilon_x}(\eta^i)} \psi(x,t)\delta(x;\mathcal{O}_{\varepsilon_x}(\eta^i))\delta(t;\mathcal{O}_{\varepsilon_t}(\tau_s))dxdt \cdot$$

$$\cdot \sum_{j=1}^{N_o} k^{ij} \left[\int_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint_{\mathcal{O}_{\varepsilon_x}(\xi^j)} \langle \mathrm{grad}_x u(\hat{x},\hat{t}), \Delta\xi^j\rangle \delta(\hat{x};\mathcal{O}_{\varepsilon_x}(\xi^j))\delta(\hat{t};\mathcal{O}_{\varepsilon_t}(\tau_s))d\hat{x}d\hat{t} \right] + o(\|\Delta\xi\|_{\mathrm{R}^{2N_o}}),$$

$$\int_0^{T_1} \iint_\Omega \psi(x,t)\,\Delta_\eta V(x,t;y,\tau)\,dxdt = \tag{42}$$

$$= -\sum_{s=1}^{N_t}\sum_{i=1}^{N_c} \int_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint_{\mathcal{O}_{\varepsilon_x}(\eta^i)} \langle \mathrm{grad}_x \psi(x,t), \Delta\eta^j\rangle \delta(x;\mathcal{O}_{\varepsilon_x}(\eta^i))\delta(t;\mathcal{O}_{\varepsilon_t}(\tau_s))dxdt \cdot$$

$$\cdot \sum_{j=1}^{N_o} \left[\int_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint_{\mathcal{O}_{\varepsilon_x}(\xi^j)} u(\hat{x},\hat{t})\delta(\hat{x};\mathcal{O}_{\varepsilon_x}(\xi^j))\delta(\hat{t};\mathcal{O}_{\varepsilon_t}(\tau_s))d\hat{x}d\hat{t} - z^{ij} \right] + o(\|\Delta\eta\|_{\mathrm{R}^{2N_c}}).$$

Here $\langle\cdot,\cdot\rangle$ denotes the inner product of two-dimensional vectors.

Let us evaluate increment of a penalty terms:

$$\Delta G(y) = G(y+\Delta y) - G(y) = -2\sum_{s=1}^{N_t}\sum_{i=1}^{N_c} g_i^+(\tau_s;y)\,\mathrm{sgn}\left(g_i^0(\tau_s;y)\right)\cdot \tag{43}$$

$$\cdot\left(\sum_{j=1}^{N_o} \left[\int_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint_{\mathcal{O}_{\varepsilon_x}(\xi^j)} u(\hat{x},\hat{t})\delta(\hat{x};\mathcal{O}_{\varepsilon_x}(\xi^j))\delta(\hat{t};\mathcal{O}_{\varepsilon_t}(\tau_s))d\hat{x}d\hat{t} - z^{ij} \right] \Delta k^{ij} - \sum_{j=1}^{N_o} k^{ij}\Delta z^{ij} + \right.$$

$$+ \sum_{j=1}^{N_o} k^{ij} \left[\int_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint_{\mathcal{O}_{\varepsilon_x}(\xi^j)} \langle \mathrm{grad}_x u(\hat{x},\hat{t}), \Delta\xi^j\rangle \delta(\hat{x};\mathcal{O}_{\varepsilon_x}(\xi^j))\delta(\hat{t};\mathcal{O}_{\varepsilon_t}(\tau_s))d\hat{x}d\hat{t} \right] +$$

$$\left. + \sum_{j=1}^{N_o} k^{ij} \int_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint_{\mathcal{O}_{\varepsilon_x}(\xi^j)} \Delta_y u(\hat{x},\hat{t})\delta(\hat{x};\mathcal{O}_{\varepsilon_x}(\xi^j))\delta(\hat{t};\mathcal{O}_{\varepsilon_t}(\tau_s))d\hat{x}d\hat{t} \right) + o(\|\Delta\eta\|_{\mathrm{R}^{2N_c}}).$$

Taking into account the fact that the components of the gradient of a functional are determined by the linear part of its increment with respect to each of the components, the validity of formulas (25)−(28) follows. □

Similarly, we can obtain formulas for the components of the gradient (20) if the possible values of the powers and locations of point sources of external effect are given by discrete sets (16), (17).

Theorem 2. *The components of gradient of the functional (20), in the case when the possible values of powers and locations of local effects at the initial moment of time takes their values on discrete sets (16), (17) with probabilities (18), (19), are determined as follows:*

$$\frac{\partial J_r(y)}{\partial k^{ij}} = \sum_{i_1=1}^{N_q^1} \cdots \sum_{i_L=1}^{N_q^L} \sum_{j_1=1}^{N_\theta^1} \cdots \sum_{j_L=1}^{N_\theta^L} \Big\{ -\sum_{s=1}^{N_t} \Big[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\eta^i)} \psi(x,t)\delta(x;\mathcal{O}_{\varepsilon_x}(\eta^i))\delta(t;\mathcal{O}_{\varepsilon_t}(\tau_s))dxdt +$$
$$+2rg_i^+(\tau_s;y)\mathrm{sgn}(g_i^0(\tau_s;y))\Big]\Big[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\xi^j)} u(x,t)\delta(\hat{x};\mathcal{O}_{\varepsilon_x}(\xi^j))\delta(\hat{t};\mathcal{O}_{\varepsilon_t}(\tau_s))d\hat{x}d\hat{t} - z^{ij}\Big] +$$
$$+2\varepsilon_1(k^{ij} - \hat{k}^{ij})\Big\} p_{Q^1}^{i_1} \cdots p_{Q^L}^{i_L} p_{\Theta^1}^{j_1} \cdots p_{\Theta^L}^{j_L}$$

$$\frac{\partial J_r(y)}{\partial z^{ij}} = \sum_{i_1=1}^{N_q^1} \cdots \sum_{i_L=1}^{N_q^L} \sum_{j_1=1}^{N_\theta^1} \cdots \sum_{j_L=1}^{N_\theta^L} \Big\{ \sum_{s=1}^{N_t} \Big[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\eta^i)} \psi(x,t)\delta(x;\mathcal{O}_{\varepsilon_x}(\eta^i))\delta(t;\mathcal{O}_{\varepsilon_t}(\tau_s))dxdt +$$
$$+2rg_i^+(\tau_s;y)\mathrm{sgn}(g_i^0(\tau_s;y))\Big] k^{ij} + 2\varepsilon_2(z^{ij} - \hat{z}^{ij})\Big\} p_{Q^1}^{i_1} \cdots p_{Q^L}^{i_L} p_{\Theta^1}^{j_1} \cdots p_{\Theta^L}^{j_L},$$

$$\frac{\partial J_r(y)}{\partial \xi_\gamma^j} = \sum_{i_1=1}^{N_q^1} \cdots \sum_{i_L=1}^{N_q^L} \sum_{j_1=1}^{N_\theta^1} \cdots \sum_{j_L=1}^{N_\theta^L} \Big\{ -\sum_{s=1}^{N_t}\sum_{i=1}^{N_c} \Big[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\eta^i)} \psi(x,t)\delta(x;\mathcal{O}_{\varepsilon_x}(\eta^i))\delta(t;\mathcal{O}_{\varepsilon_t}(\tau_s))dxdt +$$
$$+2rg_i^+(\tau_s;y)\mathrm{sgn}(g_i^0(\tau_s;y))\Big] k^{ij} \Big[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\xi^j)} \frac{\partial u(\hat{x},\hat{t})}{\partial \hat{x}_\gamma}\delta(\hat{x};\mathcal{O}_{\varepsilon_x}(\xi^j))\delta(\hat{t};\mathcal{O}_{\varepsilon_t}(\tau_s))d\hat{x}d\hat{t}\Big] +$$
$$+2\varepsilon_3(\xi_\gamma^j - \hat{\xi}_\gamma^j)\Big\} p_{Q^1}^{i_1} \cdots p_{Q^L}^{i_L} p_{\Theta^1}^{j_1} \cdots p_{\Theta^L}^{j_L},$$

$$\frac{\partial J_r(y)}{\partial \eta_\gamma^i} = \sum_{i_1=1}^{N_q^1} \cdots \sum_{i_L=1}^{N_q^L} \sum_{j_1=1}^{N_\theta^1} \cdots \sum_{j_L=1}^{N_\theta^L} \Big\{ -\sum_{s=1}^{N_t} \Big[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\eta^i)} \frac{\partial \psi(x,t)}{\partial x_\gamma}\delta(x;\mathcal{O}_{\varepsilon_x}(\eta^i))\delta(t;\mathcal{O}_{\varepsilon_t}(\tau_s))dxdt\Big] \cdot$$
$$\cdot \sum_{j=1}^{N_o} k^{ij} \Big[\int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\xi^j)} u(x,t)\delta(\hat{x};\mathcal{O}_{\varepsilon_x}(\xi^j))\delta(\hat{t};\mathcal{O}_{\varepsilon_t}(\tau_s))d\hat{x}d\hat{t} - z^{ij}\Big] +$$
$$+2\varepsilon_4(\eta_\gamma^i - \hat{\eta}_\gamma^i)\Big\} p_{Q^1}^{i_1} \cdots p_{Q^L}^{i_L} p_{\Theta^1}^{j_1} \cdots p_{\Theta^L}^{j_L},$$

where $i = 1,\dots,N_c$, $j = 1,\dots,N_o$, $\gamma = 1,2$, and $\psi(x,t)$ is solution of the initial-boundary value problem (29)–(31).

4 Results of numerical experiments

We present the results of numerical experiments obtained by solving the problem (1)−(4) under the following values of the data involved in the statement of the problem:

$$\Omega = \{x \in \mathrm{R}^2 : \quad 0 \le x_i \le 1, \quad i = 1,2\}, \quad T_f = 3, \quad \Delta T = 0.3, \quad L = 2,$$

$$a = 1, \quad \lambda = 0.001, \quad N_c = 2, \quad N_o = 3, \quad \underline{\vartheta_i} = -0.05, \ \overline{\vartheta^i} = 0.05, \ i = \overline{1,2},$$
$$N_t = 10, \quad \tau_s = 0.3s, \quad s = 1, \ldots, N_t,$$
$$\Omega_o^1 = \{x \in \Omega : \quad 0.12 \le x_1 \le 0.32, \quad 0.28 \le x_2 \le 0.48\},$$
$$\Omega_o^2 = \{x \in \Omega : \quad 0.71 \le x_1 \le 0.89, \quad 0.05 \le x_2 \le 0.22\},$$
$$\Omega_o^3 = \{x \in \Omega : \quad 0.44 \le x_1 \le 0.64, \quad 0.45 \le x_2 \le 0.63\},$$
$$\Omega_c^1 = \{x \in \Omega : \quad 0.58 \le x_1 \le 0.78, \quad 0.25 \le x_2 \le 0.42\},$$
$$\Omega_c^2 = \{x \in \Omega : \quad 0.19 \le x_1 \le 0.39, \quad 0.59 \le x_2 \le 0.78\},$$
$$q^1 \in Q^1 = \{0.050; \ 0.051; \ 0.052; \ 0.053\},$$
$$q^2 \in Q^2 = \{0.049; \ 0.050; \ 0.051; \ 0.052\},$$
$$\theta^1 \in \Theta^1 = \{x \in \Omega : \quad 0.22 \le x_1 \le 0.34, \quad 0.20 \le x_2 \le 0.38\},$$
$$\theta^2 \in \Theta^2 = \{x \in \Omega : \quad 0.65 \le x_1 \le 0.78, \quad 0.68 \le x_2 \le 0.81\}.$$

The values of the power of external effects have uniform distributions in Q^1 and Q^2, and their possible impact points are uniformly distributed in the given admissible domains Θ^1 and Θ^2. The sets of possible points of placement Θ^ν and the values of the power of external effects Θ^ν we approximate by the following discrete sets of the points:

$$\Theta^1 = \big\{(0.26; 0.28), \quad (0.27; 0.29); \quad (0.25; 0.25); \quad (0.32; 0.35)\big\},$$
$$\Theta^2 = \big\{(0.75; 0.75), \quad (0.70; 0.78); \quad (0.75; 0.79); \quad (0.76; 0.80)\big\}.$$

Each of the discrete sets consists of four elements, and the probabilities of obtaining these values, taking into account the uniformity of probability for continuous sets, are equal

$$p^\gamma_{Q^1} = p^\gamma_{Q^2} = 0.25, \quad i = 1, \ldots, 4,$$
$$p^\gamma_{\Theta^1} = p^\gamma_{\Theta^2} = 0.25, \quad j = 1, \ldots, 4.$$

In this case, instead of functional (22), taking into account (20), the following functional will be used

$$J_r(y) = \frac{1}{256} \sum_{i_1=1}^{4} \sum_{i_2=1}^{4} \sum_{j_1=1}^{4} \sum_{j_2=1}^{4} \tilde{I}_r \left(y; q^{i_1} q^{i_2} \theta^{j_1} \theta^{j_2}\right).$$

The optimized vector of parameters of feedback control under above selected parameters of the problem has dimension of $N = 22$. The sets of possible placements of the stabilizers Ω_c^i and the sensors Ω_o^j , as it is usual in practical applications, are inside the membrane, i.e. they cannot be located on the boundary Γ.

Let us describe the general scheme for implementing the iterative procedure (24) to minimize the functional (22) using the methods of the penalty function and gradient projection.

Because of the admissible domains of the parameters ξ and η are rectangular, the operators of projection onto these domains are obvious and have a simple form (Vasil'ev, 2002).

For each value of the penalty coefficient r, the regularization of the functional was carried using well-known schemes (Vasil'ev, 2002). In this case, the parameters of regularization were changed three times, namely, under the initial value $\varepsilon = 0.1$, it was decreased by 5 times, and we made $\tilde{y}$ equal the optimal value of y that was obtained at the previous step. The initial value of the penalty coefficient was set equal to 5, which increased by 5 times at each subsequent stage. These stages were carried out until the value of the main functional of problem (14), (15) obtained at two successive stages differed by more than 0.005.

Table 1: The resulting test problem solutions from two different start points y_1^0 and y_2^0.

	n	K	Z	$\xi = \left(\xi^1, \xi^2, \xi^3\right)$	$\eta = \left(\eta^1, \eta^2\right)$	$J(y)$
y_1^0	0	-0.541 -0.841 0.575 -0.826 0.848 0.375	-0.008 -0.008 0.009 -0.004 0.002 0.003	0.1418 0.2914 0.8724 0.1008 0.6332 0.6028	0.7608 0.2631 0.3725 0.6045	0.9947
	3	0.299 0.382 -0.257 0.553 0.108 -0.366	-0.023 -0.033 0.027 -0.317 0.132 0.178	0.1378 0.3858 0.8519 0.0873 0.4986 0.4500	0.7256 0.2500 0.3323 0.7800	0.0880
	6	0.296 0.332 -0.198 0.287 0.442 -0.222	0.005 0.011 -0.005 0.007 0.007 -0.005	0.2180 0.3882 0.7591 0.1923 0.5175 0.4744	0.6709 0.3036 0.2840 0.6804	0.0016
	10	0.307 0.349 -0.210 0.336 0.387 -0.249	0.003 0.008 -0.003 0.003 0.007 -0.003	0.2059 0.3967 0.7639 0.1825 0.5215 0.4707	0.6734 0.2997 0.2968 0.6893	0.0001
y_2^0	0	-0.041 -0.418 0.746 -0.259 -0.752 0.385	0.401 0.281 0.301 0.813 -0.207 0.300	0.311 0.291 0.725 0.215 0.469 0.622	0.752 0.414 0.202 0.755	2.1588
	3	-0.522 -0.173 0.278 -0.491 -0.184 0.308	-0.027 -0.001 0.041 0.050 -0.025 0.035	0.320 0.280 0.811 0.068 0.618 0.469	0.602 0.412 0.390 0.645	0.0059
	6	-0.526 -0.183 0.280 -0.492 -0.195 0.313	0.008 0.017 0.022 0.042 -0.026 0.046	0.339 0.295 0.840 0.084 0.640 0.463	0.601 0.393 0.374 0.632	0.0001

To solve two-dimensional direct and adjoint initial-boundary value problems (12), (2), (3) and (29)−(31), we used the variable directions method (Samarskii, 2001), which leads to the solution of one-dimensional loaded problems. To solve the loaded initial-boundary value problems we used an implicit finite-difference approximation scheme which was studied in Alikhanov et al. (2014). To solve the finite-difference approximated initial-boundary value problems, the numerical methods proposed in Abdullaev & Aida-zade (2014) were used. The steps for approximation by the spatial variable $h_{x_1} = h_{x_2} = 0.01$, by the time variable $h_t = 0.005$ were chosen equal.

The function $\delta\left(x; \mathcal{O}_{\varepsilon_x}(0)\right)$ was defined as the following sinusoidal-type function (Butkovskiy, 1984; Aida-zade & Bagirov, 2006):

$$\delta\left(x; \mathcal{O}_{\varepsilon_x}(0)\right) = \begin{cases} 0, & |x_1| > \sigma_{x_1} \text{ or } |x_2| > \sigma_{x_2}, \\ \prod\limits_{i=1}^{2} \frac{1}{2\sigma_{x_i}} \left[1 + \sin\left(\frac{2x+\sigma_{x_i}}{2\sigma_{x_i}}\pi\right)\right] & |x_1| \le \sigma_{x_1} \text{ and } |x_2| \le \sigma_{x_2}. \end{cases}$$

Thus, the ε_x-neighborhood of the origin point is a square with a side equal to σ_x. In numerical calculations, σ_{x_1} and σ_{x_2} were chosen to be equal to h_{x_1} and h_{x_2}, i.e. $\delta\left(x; \mathcal{O}_{\varepsilon_x}(0)\right)$ assumed nonzero values in 49 cells of the grid area adjacent to the point x. The form of the function $\delta\left(x; \mathcal{O}_{\varepsilon_x}(0)\right)$ ensures to say functional $J_r(y)$ is smooth along the optimized coordinates ξ^j, η^i, $i = 1, \ldots, N_c$, $j = 1, \ldots, N_o$ (Aida-zade & Bagirov, 2006).

The neighborhood of time τ was determined by the time interval $\mathcal{O}_{\varepsilon_t}(\tau) = [\tau - 2h_t, \tau + h_t]$, and the function $\mathcal{O}_{\varepsilon_t}(\tau)$ is defined as a continuous asymmetric triangular-like function:

$$\delta\left(t; \mathcal{O}_{\varepsilon_t}(\tau_s)\right) = \begin{cases} 0, & t \notin \mathcal{O}_{\varepsilon_t}(\tilde{\tau}), \\ (t + (2h_t - \tilde{\tau}))/3h_t^2 & t \in [\tilde{\tau} - 2h_t, \tilde{\tau}], \\ (-2t + 2(\tilde{\tau} - h_t))/3h_t^2 & t \in [\tilde{\tau}, \tilde{\tau} + h_t], \end{cases}$$

It is easy to check that

$$\int\limits_0^{T_f} \delta(t; \mathcal{O}_{\varepsilon_t}(\tilde{\tau})) = \int\limits_{\mathcal{O}_{\varepsilon_t}(\tilde{\tau})} \delta(t; \mathcal{O}_{\varepsilon_t}(\tilde{\tau})) = 1.$$

Tables 1, 2 present the results of calculations in which two values y_1^0, y_2^0 were used as the initial approximation for the iterative process. Table 1 presents the values of the matrices k and z in the following order: $(k^{11}, \ldots, k^{1N_o}, \ldots, k^{N_c1}, \ldots, k^{N_cN_o})$, $(z^{11}, \ldots, z^{1N_o}, \ldots, z^{N_c1}, \ldots, z^{N_cN_o})$.

Table 1 presents the values of the results of ten iterations of the problem solution which were obtained using the above two initial approximations.

Table 2: Solutions of the test problem obtained from the first initial point y_1^0 at the values of error in measurements of 1%, 2%, 5%.

y_1	n	K	Z	$\xi = \left(\xi^1, \xi^2, \xi^3\right)$	$\eta = \left(\eta^1, \eta^2\right)$	$J(y)$
1%	0	-0.541 -0.841 0.575 -0.826 0.848 0.375	-0.008 -0.008 0.009 -0.004 0.002 0.003	0.142 0.291 0.872 0.101 0.633 0.603	0.761 0.263 0.373 0.605	0.9954
	10	0.307 0.349 -0.210 0.337 0.386 -0.249	0.003 0.007 -0.003 0.002 0.008 -0.002	0.205 0.397 0.763 0.182 0.521 0.471	0.673 0.300 0.297 0.689	0.0001
2%	0	-0.541 -0.841 0.575 -0.826 0.848 0.375	-0.008 -0.008 0.008 -0.004 0.002 0.003	0.142 0.291 0.872 0.101 0.633 0.603	0.761 0.263 0.373 0.605	0.9958
	10	0.306 0.348 -0.210 0.335 0.388 -0.249	0.003 0.008 -0.003 0.002 0.008 -0.002	0.206 0.397 0.764 0.182 0.521 0.471	0.674 0.300 0.296 0.689	0.0001
5%	0	-0.541 -0.841 0.575 -0.826 0.848 0.375	-0.008 -0.008 0.009 -0.003 0.002 0.003	0.142 0.291 0.872 0.101 0.633 0.603	0.761 0.263 0.373 0.605	0.9969
	10	0.307 0.348 -0.210 0.336 0.387 -0.249	0.003 0.008 -0.003 0.003 0.007 -0.003	0.206 0.397 0.764 0.183 0.522 0.471	0.673 0.299 0.296 0.689	0.0001

It can be seen that, as mentioned above, due to the possible multi-extremity of objective functional, the results of optimization obtained from different starting points differ in arguments, although the difference by functionality is not significant. Here it is also necessary to take into account (as other specially conducted numerical experiments showed) that the functional of the problem has a strong ravine structure.

Computer experiments were carried out to observe the process of damping oscillations under optimal values of the synthesized feedback parameters under the assumption that measurements were made with interferences, namely:

$$\tilde{u}_s^j = \int\limits_{\mathcal{O}_{\varepsilon_t}(\tau_s)} \iint\limits_{\mathcal{O}_{\varepsilon_x}(\xi^j)} u(x,t)[1+\kappa^j(t)]\delta(x;\mathcal{O}_{\varepsilon_x}(\eta^i))\delta(t;\mathcal{O}_{\varepsilon_t}(\tau_s))dxdt, \quad j=1,\dots,N_o, \quad s=1,\dots,N_t.$$

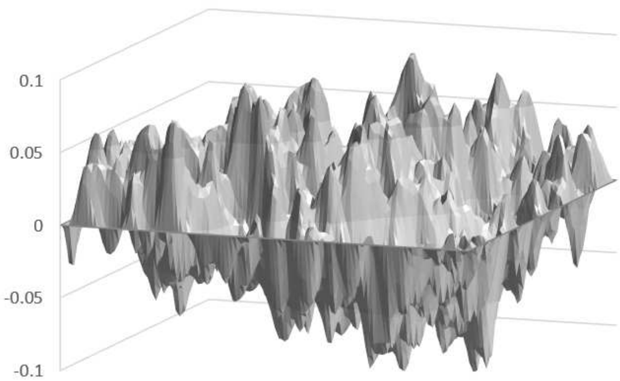
Here $\kappa^j(t)$ for each t is a random variable uniformly distributed on the interval $[-\zeta;\zeta]$. In the performed experiments the values of ζ were chosen to be 0.01; 0.02; 0.05, what corresponded to a measurement error of 1%, 2% and 5% of the measured value.

Table 2 presents the results obtained at six iterations of the solution for the synthesis of feedback parameters in the presence of an error in the measurements carried out. As can be seen from the comparison of the obtained values of the feedback parameters, tentatively they differ in proportion to the errors of the measurements.

An important indicator of the quality of controlling the damping process with feedback parameters y is the function:

$$E(T;y^*) = \int\limits_Q \iint\limits_\Theta \Big[\int\limits_T^{T+\Delta T} \iint\limits_\Omega \mu(x)\big[u(x,t;y^*,q,\theta)\big]^2 dxdt \Big] \rho_\Theta(\theta)\rho_Q(q)d\theta dq, \quad T \ge 0.$$

The function characterizes numerically the result of the process control for all possible values of external effects by average. Figure 1 presents the graphs of the function $E(T;y^*)$ obtained under optimal feedback parameters y^* and $T_f = 3$ at levels of error of measurements equal to 0% (without error), 1%, 2% and 5%. At these graphs it can be seen that the quality of the stabilization process control corresponds to the value of error of measurements of state process which were carried out. Figure 2 presents a graph of the function $u(x,t) = u(x,t;y^*,q,\theta)$, $x \in \Omega$, which defines the state of the membrane at the time when the control process ends at $t = T_f = 3$ and $q_1 = 0.52$, $q_1 = 0.49$, $\theta_1 = (0.25, 0.25)$, $\theta_2 = (0.75, 0.75)$.



H

Figure 1: Graphs of function $u(x, T)$ at $T = 3$.

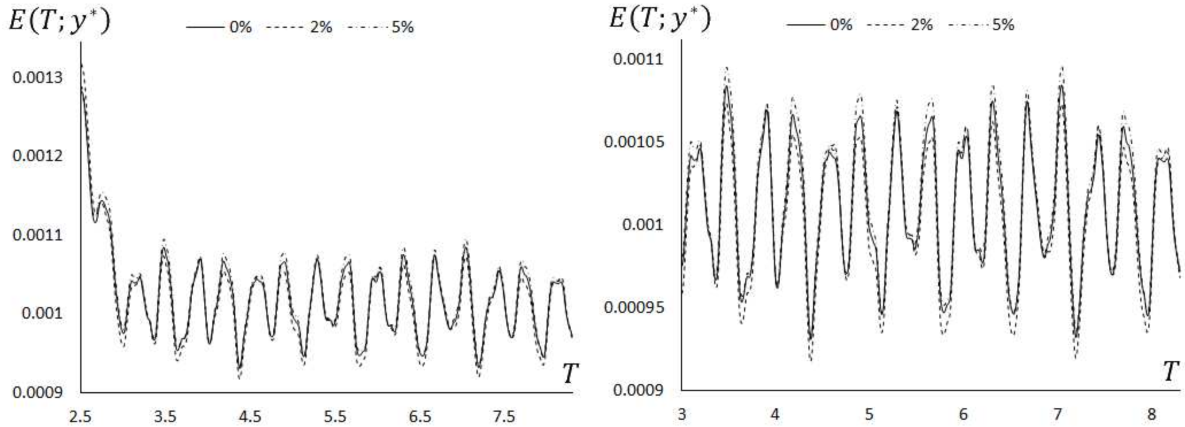


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Figure 2: Graphs of function $E(T, y^*)$ at different errors of $u(x, T)$ at $T = 3$.

5 Conclusion

An approach to the optimal synthesis of lumped control effects in systems with distributed parameters is proposed in this paper. As an example, the problem of synthesis of the control of lumped stabilizers while damping membrane oscillations is considered. The modes of functioning of stabilizers are defined by a linear dependence on measurements of the membrane state at the neighborhood of the measurement points. The optimized parameters of the problem are: 1) the parameters of linear feedback which determines the stabilizer operating modes; 2) the placements of the stabilizers; 3) the placements of the points of measurement of the membrane state.

The problem under consideration is reduced to the parametric problem of optimal control of a system with distributed parameters. The formulas for the gradient of the functional of the problem in the space of synthesized parameters are obtained. The formulas made it possible to use efficient first-order optimization methods for the numerical solution of the synthesis problem.

The paper presents the results of numerical experiments and analyzes the effect of measurement errors on the process of membrane stabilization.

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